Math 4200
Wednesday September 9

1.5: From last weeks notes: Review of chain rule for curves and CR equations; precise discussion of the differential map in our context, which is also used more generally for differentiable transformations between Euclidean spaces and differentiable manifolds; new in today's notes: proofs of the full CR theorem and inverse function theorem using Math 3220 ideas.

Announcements:

Warmup exercise: Recall the CR equations for the partial derivatives of the real and imaginary parts of a complex differentiable function $f(z)$:

$$f(x+iy) == u(x,y) + i\,v(x,y)$$

$$CR \begin{cases} u_x = v_y \\ v_x = -u_y \end{cases}$$

Re-derive the CR equations using the chain rule for curves and the identity

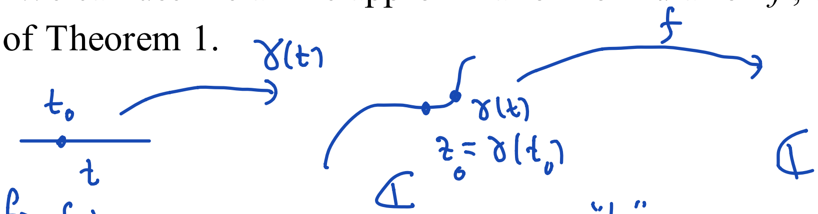
$$f'(z) = f_x = \frac{1}{i} f_y$$

Friday review

2) <u>Theorem</u> (Chain rule for curves) If f is differentiable at z_0 and $\gamma: I \subseteq \mathbb{R} \to \mathbb{C}$ is a parametric curve $\gamma(t) = x(t) + i\,y(t)$ such that $\gamma(t_0) = z_0$ and such that $\gamma'(t_0) = x'(t_0) + i\,y'(t_0)$ exists, then

$$(f \circ \gamma)'(t_0) = \underbrace{f'(\gamma(t_0))}\,\underbrace{\gamma'(t_0)}$$

<u>proof</u> We can use the affine approximation formula for f, at $\gamma(t_0)$, and mimic the proof of Theorem 1.



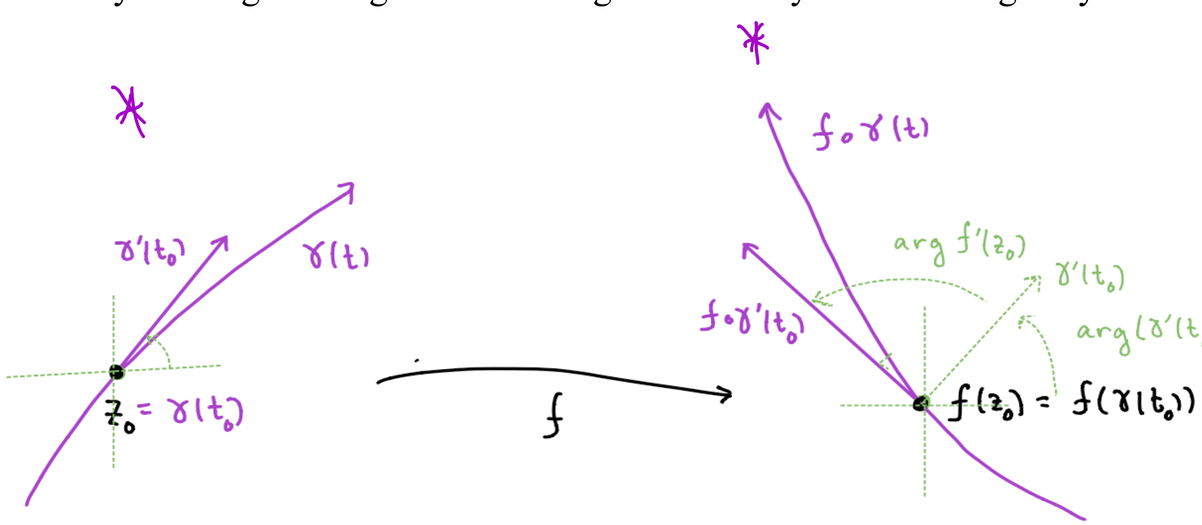
approx for f:

$$f(\gamma(t)) = f(\gamma(t_0)) + f'(\gamma(t_0))\overbrace{(\gamma(t)-\gamma(t_0))}^{\text{"}h\text{"}} + h\,\varepsilon(h) \qquad f \text{ diff'ble @ } z_0$$

$$\frac{f(\gamma(t)) - f(\gamma(t_0))}{t-t_0} = f'(\gamma(t_0))\,\frac{\gamma(t)-\gamma(t_0)}{t-t_0} + \frac{h\,\varepsilon(h)}{t-t_0} \to \gamma'(t_0)\cdot 0 = 0$$

$\lim_{t \to t_0}$: $\boxed{(f\circ\gamma)'(t_0) = f'(\gamma(t_0))\,\gamma'(t_0) + 0}$

Domain-range geometry implied by the chain rule for curves. Consider the curve $\gamma(t)$ which has image in the domain of f, along with the curve $f \circ \gamma(t)$ which has image in the range of f. Let $f'(\gamma(t_0)) = r\,e^{i\theta}$. Then the image curve tangent vector is obtained by rotating the original curve tangent vector by r and scaling it by θ.



$$(f\circ\gamma)'(t) = \underbrace{f'(\gamma(t))}\,\gamma'(t)$$

@ t_0 $\qquad \underbrace{|f'(z_0)|}_{\text{scaled by mod.}}\;\underbrace{e^{i\arg f'(z_0)}\,\gamma'(t_0)}_{\text{rotated by arg } f'(z_0)}$

Friday notes

Conformal transformations and differential map discussion:

(i) The precise definition of the *tangent space* at $z_0 \in \mathbb{C}$ is the set of all *tangent vectors* there, i.e. tangent vectors to curves passing through z_0:

$$T_{z_0}\mathbb{C} := \left\{ \gamma'(t_0) \,\middle|\, \gamma \text{ is differentiable at } t_0 \text{ and } \gamma(t_0) = z_0 \right\}$$

(ii) If $f(z)$ is a function from $\mathbb{C}$ to $\mathbb{C}$ that arises from a real-differentiable function $F : A \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, then the *differential of f at z_0* is defined by

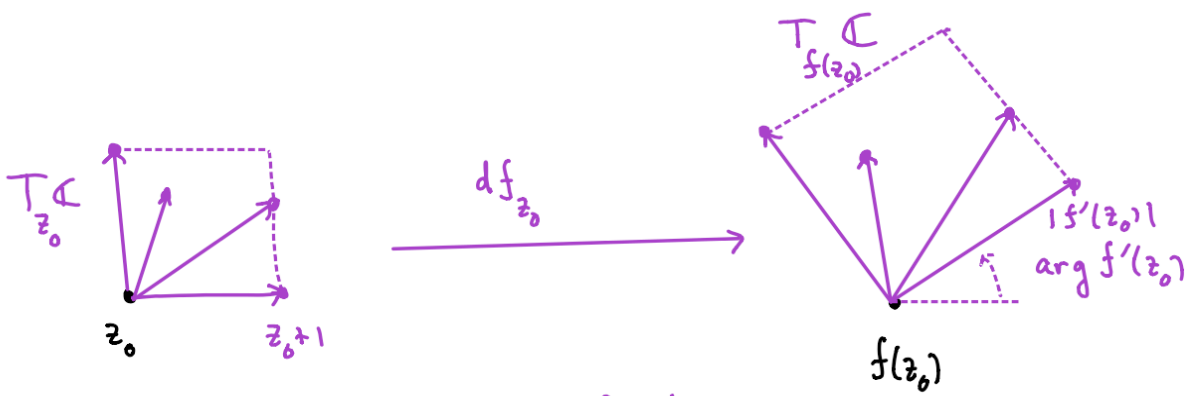
$$df_{z_0}(\gamma'(t_0)) := (f \circ \gamma)'(t_0).$$

$$df_{z_0} : T_{z_0}\mathbb{C} \to T_{f(z_0)}\mathbb{C}.$$

(iii) By the chain rule for curves, *if* $f(z)$ is complex differentiable at z_0, then

$$df_{z_0}(\gamma'(t_0)) := (f \circ \gamma)'(t_0) = f'(z_0)\gamma'(t_0)$$

Geometrically, this means that for complex differentiable functions f, the differential map is a linear transformation from $T_{z_0}\mathbb{C}$ to $T_{f(z_0)}\mathbb{C}$ which is a rotation-dilation.



$$df(\gamma'(t_0)) = f \circ \gamma'(t_0)$$
$$= f'(z_0)\gamma'(t_0)$$

(iv) A function $f : \mathbb{C} \to \mathbb{C}$ is called *conformal* at z_0 iff its differential transformation preserves angles between tangent vectors. Since rotation-dilations have this property, a function f which is complex differentiable at z_0, and for which $f'(z_0) \neq 0$, is conformal at z_0. (It turns out that if f is conformal at z_0 and also preserves orientations of pairs of tangent vectors, then f is complex differentiable at z_0.)

Illustration. Consider

$$f(z) = z^2,\ z_0 = 1 + i,$$

$$f(z_0) = 2\,i,\ f'(z_0) = 2 + 2\,i = 2\sqrt{2}\,e^{i\frac{\pi}{4}}$$

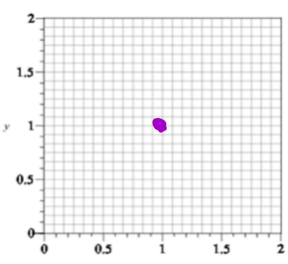
Below, are parts of a rectangular coordinate grid in the domain, and the image of that grid in the range space.

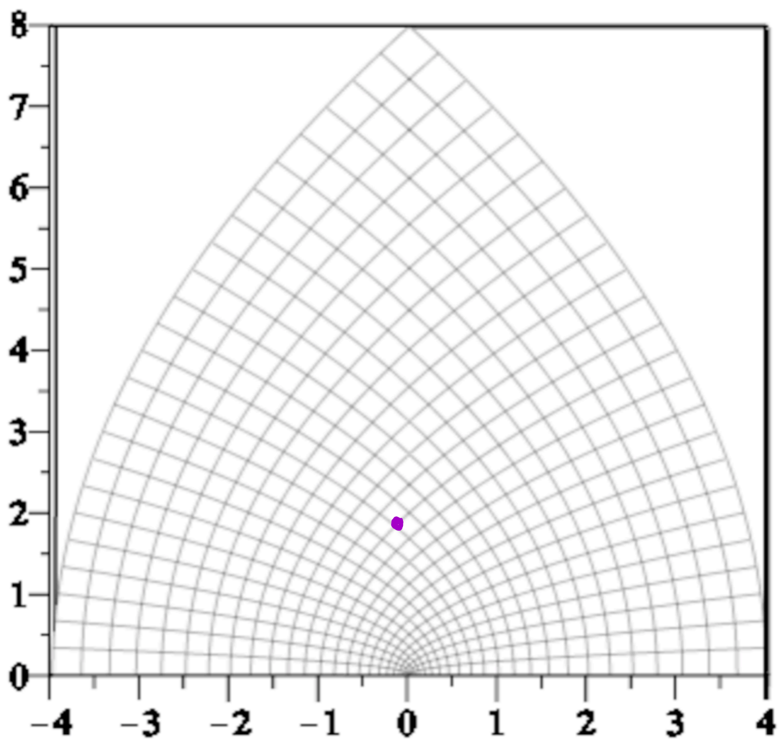
a) Why are the images of the real and imaginary grid lines also perpendicular?
b) Find the formula for the differential map

$$df_{z_0} : T_{z_0}\mathbb{C} \rightarrow T_{f(z_0)}\mathbb{C}$$

and illustrate the rotation dilation.

finish discussion on Wed. next week.





The next two theorems are applications of Math 3220 analysis and the correspondence between $f: A \subseteq \mathbb{C} \to \mathbb{C}$, and $F: A \subseteq \mathbb{R}^2 \to \mathbb{R}^2$: For each

$$f(x+iy) = u(x, y) + i\, v(x, y)$$
$$f: A \subseteq \mathbb{C} \to \mathbb{C},\ A \text{ open}$$

there corresponds

$$F(x, y) = (u(x, y),\ v(x, y))$$
$$F: A \subseteq \mathbb{R}^2 \to \mathbb{R}^2,\ A \text{ open}$$

<u>Theorem</u> (full CR Theorem) Let $A \subseteq \mathbb{C}$ open, $f: A \to \mathbb{C}$, $z_0 \in A$. Write

$$f(z) = f(x + i\,y) = u(x, y) + i\, v(x, y)$$

where

$$u(x, y) = \mathrm{Re}(f(x+i\,y),\ v(x, y) = \mathrm{Im}(f(x+i\,y)$$

Then if f is complex differentiable at $z_0 = x_0 + i\,y_0$ *if and only if* the following two conditions hold:

(1) The *Cauchy-Riemann equations* hold at (x_0, y_0):

$$u_x(x_0, y_0) = v_y(x_0, y_0)$$
$$u_y(x_0, y_0) = -v_x(x_0, y_0);$$

AND

(2) $F(x, y) = (u(x, y),\ v(x, y))$ is *Real differentiable* at (x_0, y_0) in the affine approximation sense you discussed in Math 3220. In particular real differentiability is implied by the condition that all of the partial derivatives u_x, y_y, v_x, v_y exist and are continuous in a neigborhood of (x_0, y_0).

proof:

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$f(z) = u(x,y) + iv(x,y)$

$f'(z_0) = a+bi$ exists

$\Leftrightarrow$ $f(z_0+h) = f(z_0) + (a+bi)h + h\,\varepsilon(h)$

s.t. $\lim_{h\to 0} \varepsilon(h) = 0$

write $z = z_0+h$,

$$\Leftrightarrow \lim_{z\to z_0} \frac{|f(z)-f(z_0)-(a+bi)(z-z_0)|}{|z-z_0|} = 0$$

$$\Leftrightarrow \lim_{(x+iy)\to(x_0+iy_0)} \frac{\left| u(x,y)+iv(x,y)-u(x_0)-iv(x_0,y_0) - (a+bi)(x-x_0+i(y-y_0)) \right|}{|(x-x_0)+i(y-y_0)|} = 0$$

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$F(x,y) = (u(x,y), v(x,y))$

$$\Leftrightarrow \lim_{(x,y)\to(x_0,y_0)} \frac{\left\| \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} - \begin{bmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{bmatrix} - \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} \right\|}{\left\| \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \right\|} = 0$$

$\Leftrightarrow$ F is real-diff'ble @ (x_0,y_0) with differential matrix @ (x_0,y_0):

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

∎

Theorem (Inverse function theorem) Let f be complex differentiable in a neighborhood of z_0, with $f'(z_0) \neq 0$ and $f'(z)$ continuous. Then there exist open sets U, V with $z_0 \in U$, $f(z_0) \in V$ such that $f: U \to V$ is a bijection and $f^{-1}: V \to U$ is also analytic. Furthermore

$$(f^{-1})'(f(z)) = \frac{1}{f(z)}$$

$\forall\, z \in U$.

proof:

$\mathbb{C}$	$\mathbb{R}^2$
$f: A \subset \mathbb{C} \to \mathbb{C}$	$F: A \subset \mathbb{R}^2 \to \mathbb{R}^2$

f analytic in a nbhd A of z_0, $f'(z)$ continuous on A, $f'(z_0) \neq 0$,

$$f(z) = f(x+iy) = u(x,y) + iv(x,y)$$

$$f'(z_0) = a + bi = f_x(z_0) = -i f_y(z_0)$$

$$\left(\begin{aligned} a &= u_x(x_0,y_0) = v_y(x_0,y_0) \\ b &= v_x(x_0,y_0) = -u_y(x_0,y_0) \end{aligned} \right)$$

$\overset{1}{\Longrightarrow}$

$F(x,y) = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$ real differentiable in a nbhd of (x_0, y_0)

$$DF(x,y) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

$$DF(x_0,y_0) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad \text{is invertible} \quad (\det = a^2 + b^2 = |f'(z_0)|^2)$$

$\overset{2}{\Downarrow}$ 3220 inverse fcn thm

$\exists\, \mathcal{U}, V$ open, $(x_0,y_0) \in \mathcal{U} \subset A$; $F(x_0,y_0) \in V$
s.t. $F: \mathcal{U} \to V$ is a bijection, and
$F^{-1}: V \to \mathcal{U}$ is real differentiable.
Furthermore, by the 3220 chain rule and since

$$F^{-1}(F(x,y)) = (x,y) \quad \forall (x,y) \in \mathcal{U}$$

the matrix product of derivative matrices

$$DF^{-1}(F(x,y))\, DF(x,y) = I \leftarrow \text{identity matrix}$$

In particular,

$$DF^{-1}(F(x,y)) = [DF(x,y)]^{-1} \quad \forall (x,y) \in \mathcal{U}.$$

Note, $DF(x,y) = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$, so

$$DF^{-1}(F(x,y)) = \frac{1}{\alpha^2 + \beta^2} \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

is also a rotation-dilation

$\overset{3}{\Longleftarrow}$

$\exists\, \mathcal{U}, V$ open, $z_0 \in A \subset \mathcal{U}$, $f(z_0) \in V$
s.t. $f: \mathcal{U} \to V$ is a bijection,
and $f^{-1}: V \to \mathcal{U}$
has real differentiable counterpart F^{-1} whose partials satisfy CR eqtns

$\Rightarrow f^{-1}$ is analytic on V.

And since $f^{-1}(f(z)) = z$

$$(f^{-1})'(f(z)) \cdot f'(z) = 1$$

$$(f^{-1})'(f(z)) = \frac{1}{f'(z)}, \quad \forall z \in \mathcal{U}.$$

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Loose end: (applies to the hw problem 1.5.16)

Theorem Let A be an open connected set in $\mathbb{C}$, $f: A \to \mathbb{C}$ analytic, with $f'(z) = 0 \ \forall \ z \in A$. Then f is constant.

proof: For open sets, *connected* and *path-connected are equivalent.* Any continuous path connecting two points in A can be approximated with a continuously differentiable (C^1) path connecting the same two points. Let z_0 be any fixed point in A. Let $z \in A$ be any other point. Let γ be a C^1 curve,

$$\gamma : [a, b] \to A$$
$$\gamma(a) = z_0$$
$$\gamma(b) = z$$

Then by the fundamental theorem of Calculus (applied to the real and imaginary parts of f),

$$\begin{aligned} f(z) - f(z_0) &= \int_a^b \frac{d}{dt} f(\gamma(t)) \ dt \\ &= \int_a^b f'(\gamma(t)) \ \gamma'(t) \ dt \\ &= \int_a^b 0 \ dt = 0. \end{aligned}$$

QED

(Or, you showed in Math 3220 that a continuously differentiable function of several variables defined on an open connected set and with all partial derivatives equal to zero, is constant. That theorem applies here, since the partials of $\mathrm{Re}(f)$, $\mathrm{Im}(f)$ are zero if $f' \equiv 0$.)